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Math 132: Differential Topology

§ Inverse Function Theorem and immersions

- Suppose M, N are smooth manifolds of the same dimension and $f: M \to N$ is a smooth map.

  We say f is a local diffeomorphism at $p \in M$ if f maps any sufficiently small open neighborhood U of p diffeomorphically onto an open set $f(U)$.

  Clearly (from last lecture), if f is a local diffeo. at $p \in M$, then $df_p : T_pM \to T_{f(p)}N$ is an isomorphism.

  The converse is also true!

Thm (Inverse Function Theorem)

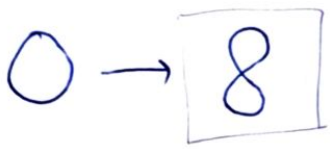
If $f: M \to N$ is a smooth map whose derivative df_p at $p \in M$ is an isomorphism, then f is a local diffeomorphism at p.

proof) Easily follows from the Inverse Function Theorem for open subsets of Euclidean spaces, using local parametrizations. ■

Morally: local behavior of smooth maps are completely determined by their derivatives.

- More generally, when $\dim M \leq \dim N$, a map $f : M \to N$ is said to be an immersion at $p \in M$ if df_p is injective. If it is an immersion at every point, we simply call it an immersion.

Example: ① $S^1 \to \mathbb{R}^2$ ② $\mathbb{R} \to T^2$ with irrational slope



(The image may not be a manifold.)

Thm (local immersion theorem)

If $f : M \to N$ is an immersion at $p \in M$, then there exist local coordinates around $p \in M$ and $f(p) \in N$ such that

$$f(x_1, \cdots, x_m) = (x_1, \cdots, x_m, 0, \cdots, 0).$$

proof) Choose any local parametrizations

$$\phi : U \longrightarrow M, \quad U \subset \mathbb{R}^m,$$
$$0 \longmapsto p$$
$$\psi : V \longrightarrow N, \quad V \subset \mathbb{R}^n.$$
$$0 \longmapsto f(p)$$

By making U smaller if necessary, we may assume $f \circ \phi(U) \subset \psi(V)$ to get a commutative diagram

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ U \arrow[u, "\phi"] \arrow[r, "g"'] & V \arrow[u, "\psi"'] \end{tikzcd}, \quad g := \psi^{-1} \circ f \circ \phi.$$

By the assumption, $dg_0 : \mathbb{R}^m \to \mathbb{R}^n$ is injective.

Hence, by change of basis in $\mathbb{R}^n$, we may assume $dg_0 = \begin{array}{c} \overbrace{}^{m} \\ \left(\begin{array}{c} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \\ \hline 0 \end{array} \right) \end{array} \begin{array}{l} \}m \\ \}n-m \end{array}$.

Now define $G : U \times \mathbb{R}^{n-m} \longrightarrow \mathbb{R}^n$ by

$$(x, z) \longmapsto g(x) + (0, z).$$

Then $dG_0 = \underbrace{\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}}_{n} \}n$. Therefore, by the Inverse Function Theorem,

G is a local diffeomorphism at 0.

Note also that $g = G \circ \iota$, where
$$\iota : U \longrightarrow U \times \mathbb{R}^{n-m}, \quad x \longmapsto (x, 0)$$
is the canonical immersion.

It follows that, after shrinking U and V sufficiently, we have

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ U \arrow[u, "\phi"] \arrow[r, "\iota"] & V \arrow[u, "\psi \circ G"'] \end{tikzcd}$$

, as desired. ∎

<u>Cor</u> If f is an immersion at p, then it is an immersion in a neighborhood of p.

That is, <u>immersion is an open condition</u>.

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<u>Def</u> An immersion $f: M \to N$ that is injective and proper is called an <u>embedding</u>.

(preimage of every compact set is compact; automatic when M is compact.)

<u>Thm</u> An embedding $f: M \to N$ maps M diffeomorphically onto a submanifold $f(M)$ of N.

proof) It suffices to show that the image of any open subset $W \subset M$ is an open subset of $f(M)$.

If $f(W)$ is not open in $f(M)$, there must be a sequence of points $q_i \in f(M) \setminus f(W)$ that converge to a point $q \in f(W)$.

By injectivity, each q_i has exactly one preimage $p_i \in M \setminus W$

and q —— " —— $p \in W$.

By properness, $\{p, p_i \mid i \in \mathbb{N}\}$ is compact, and hence $\{p_i\}$ must have a subsequence converging to some $p' \in \{p, p_i \mid i \in \mathbb{N}\} \subset M$.

Since $f(p') = \lim_{j \to \infty} f(p_{i_j}) = \lim_{j \to \infty} q_{i_j} = q = f(p)$, p' must be p.

However, this contradicts our assumption that W is open.

Therefore, f is an open map, and the result follows from the local immersion theorem. ■